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REMARKS ON FORMATIONS

BY
A. BALLESTER-BOLINCHES
Departamento de Algebra, Facultad de Matematicas, 46100 Burjassot (Valencia), Spain

ABSTRACT
Applying the normalizer theory of finite groups developed in 1989, we undertake some questions concerning the theory of formations of finite groups.

1. Introduction

The purpose of this note is to study some questions concerning the theory of formations of finite groups.

The introduction of the $\mathfrak{H}$-normalizers of a finite, not necessarily soluble, group associated with a Schunck class $\mathfrak{H}$ of the form $E_\Phi\mathfrak{f}$, for some formation $\mathfrak{f}$ in [1] provides some new techniques to undertake some old problems, to improve some classical results and to give alternative approaches valid not only in the soluble case but also in the general case.

Consider a local formation of finite soluble groups, $\mathfrak{f}$ say. Carter and Hawkes [5] and Doerk [7] have shown that there is a unique full and integrated local definition F of $\mathfrak{f}$. Doerk has used this definition to prove that the formula

$$f_1(p) = \left(G \in \mathfrak{S} \,\middle|\, \mathfrak{f}\text{-normalizers of } G \text{ belong to } F(p)\right) \qquad (p \text{ a prime number})$$

defines the unique maximal local definition of $\mathfrak{f}$ in the universe of all finite soluble groups.

Doerk, Semetkov and Schmid have posed the problem of whether every local formation of finite, not necessarily soluble, groups has a unique maximal local definition. The answer is negative in general. In fact, Förster and Salomon [11] have obtained local formations without a unique maximal local definition.

In section 3 we give a description of local formations with a unique maximal local definition (some preliminary observations can be found in [1]). This approach is a generalization of Doerk's one in the soluble case.

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On the other hand, Carter, Fischer and Hawkes [6] have proved that if $\mathfrak{f}$ is a saturated formation of finite soluble groups, the unique largest subgroup-closed class contained in $\mathfrak{f}$ is also a saturated formation. Salomon [14] shows that this result does not remain true in the general case. In section 3, a description of local formations $\mathfrak{f}$ such that the unique largest subgroup-closed class contained in $\mathfrak{f}$ is also a local formation is given, from which examples and counterexamples emerge.

In section 4 we study the formation generated by a finite group G in terms of the formation generated by quotients of G (of a certain type). This result can be useful in induction arguments and is an improvement of the main result in [3], which is generalized in [12].

Finally, we characterize those Schunck classes which are saturated formations by means of a property of their boundaries.

2. Preliminaries

For the purpose of greater generality, in section 3 we employ the notion of an $\mathfrak{X}$-local formation as introduced in [10].

Denote by $\mathfrak{X}_0$ the class of all finite simple groups. For any subclass $\mathfrak{X}$ of $\mathfrak{X}_0$ we put $\mathfrak{X}' = \mathfrak{X}_0 \backslash \mathfrak{X}$ and let $X(\mathfrak{X})$ be the cyclic groups in $\mathfrak{X}$. Occasionally, it will be convenient to denote the group of prime order p simply by p rather than Z_p.

An *$\mathfrak{X}$-formation function f* associated to each $X \in X(\mathfrak{X}) \cup \mathfrak{X}'$ a formation, possibly empty, $f(X)$. If f is an $\mathfrak{X}$-formation function, then the *$\mathfrak{X}$-local formation defined by f*, $\mathrm{LF}_{\mathfrak{X}}(f)$, is the class of all finite groups G satisfying the following two conditions:

(1) $\mathrm{Aut}_G(H/K) \in f(p)$ for all chief factors H/K of G such that the composition factor of H/K is an $\mathfrak{X}$-group the order of which is divisible by p, if $p \in X(\mathfrak{X})$ (i.e. $Z_p \in X(\mathfrak{X})$); and

(2) $G/L \in f(E)$ whenever G/L is a monolithic quotient of G such that the composition factor of its socle $S(G/L)$ is isomorphic to E, if $E \in \mathfrak{X}'$.

Clearly, $\mathrm{LF}_{\mathfrak{X}}(f)$ is a formation.

Any $\mathfrak{X}$-formation function g such that $\mathrm{LF}_{\mathfrak{X}}(f) = \mathrm{LF}_{\mathfrak{X}}(g)$ is called an *$\mathfrak{X}$-local definition of the local formation* $\mathrm{LF}_{\mathfrak{X}}(f)$.

Consider the $\mathfrak{X}$-local formation $\mathfrak{f} = \mathrm{LF}_{\mathfrak{X}}(f)$ and let F be the $\mathfrak{X}$-formation function defined by

$$F(X) = \begin{cases} \mathfrak{G}_p QR_0(\mathrm{Aut}_G(H/K) \mid G \in \mathfrak{f},\ H/K\ \mathfrak{X}_p\text{-chief factor of } G), \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } p \in X(\mathfrak{X}), \\ \mathfrak{f}, \qquad\qquad\qquad\qquad\qquad\quad\ \ \text{if } X \in \mathfrak{X}', \end{cases}$$

where $\mathfrak{X}_p = (X \in \mathfrak{X} \mid p \text{ divides the order of } X)$.

Clearly, F is an $\mathfrak{X}$-local definition of the local formation $LF_{\mathfrak{X}}(f)$. We say that F is the *full and integrated local definition* of $\mathfrak{f}$ (in the sense of [11], section 1).

Recall that if $\mathfrak{H}$ is a class of groups, the boundary $b(\mathfrak{H})$ of $\mathfrak{H}$ is defined by

$$b(\mathfrak{H}) = (G \in \mathfrak{G}\backslash\mathfrak{H} \mid \text{if } 1 \neq N \trianglelefteq G, \text{ then } G/N \in \mathfrak{H})$$

and $h(\mathfrak{H})$ is the class of all groups without quotients in $\mathfrak{H}$; here, and elsewhere in this note, $\mathfrak{G}$ denotes the class of all finite groups. Moreover, if $b(\mathfrak{H})$ consists of monolithic groups, denote $b_{\mathfrak{X}}(\mathfrak{H}) = (G \in b(\mathfrak{H}) \mid$ the composition factor of $\mathrm{Soc}(G)$ is an $\mathfrak{X}$-group). In particular, if p is a prime,

$$b_p(\mathfrak{H}) = (G \in b(\mathfrak{H}) \mid \mathrm{Soc}(G) \text{ is a } p\text{-group}).$$

In our discussion of $\mathfrak{X}$-local formations with a unique maximal $\mathfrak{X}$-local definition, the following definitions and results will turn out to be crucial.

(2.1) Definition [11]. Consider the $\mathfrak{X}$-local formation $\mathfrak{f} = LF_{\mathfrak{X}}(f)$ and let F be its full and integrated local definition. A group $G \in b_{\mathfrak{X}}(\mathfrak{f})$ is called *$\mathfrak{X}$-dense* with respect to $\mathfrak{f}$, if $G \in b(F(p))$ for each prime $p \in \pi(\mathrm{Soc}(G))$. Further, $b(\mathfrak{f})$ is said to be *$\mathfrak{X}$-wide*, if there does not exist an $\mathfrak{X}$-dense group $G \in b_{\mathfrak{X}}(\mathfrak{f})$.

In the final step of an induction argument there frequently appear primitive groups. Recall that a *primitive* group is a group G such that for some maximal subgroup U of G, $U_G = 1$ (where U_G is the intersection of all G-conjugates of U, i.e., the unique largest normal subgroup of G contained in U).

A primitive group is of one of the following types:

(1) $\mathrm{Soc}(G)$, the socle of G, is an abelian minimal normal subgroup of G, complemented by U.

(2) $\mathrm{Soc}(G)$ is a non-abelian minimal normal subgroup of G.

(3) $\mathrm{Soc}(G)$ is the direct product of the two minimal normal subgroups of G which are both non-abelian and complemented by U.

(2.2) Definitions [1]. (a) Let M be a maximal subgroup of a group G. Then the group $X = G/M_G$ is a primitive group; we say that M is of *type i* if $X \in \mathfrak{P}_i$ $(1 \leq i \leq 3)$, where $\mathfrak{P}_i$ denotes the class of all primitive groups of type i; and M is a *monolithic maximal subgroup* of G if M is of type 1 or type 2.

(b) Given a Schunck class $\mathfrak{H}$, a maximal subgroup U of a group G is called *$\mathfrak{H}$-normal* in G if $G/U_G \in \mathfrak{H}$, and *$\mathfrak{H}$-abnormal* otherwise.

(c) Let U, G and $\mathfrak{H}$ be as above. U is a *$\mathfrak{H}$-critical* in G, if U is a $\mathfrak{H}$-abnormal monolithic maximal subgroup of G and $G = UF\#(G)$ where $F\#(G) = \mathrm{Soc}(G \bmod \Phi(G))$.

(2.3) THEOREM [1]. *For a Schunck class $\mathfrak{H}$, the following three statements are pairwise equivalent:*

(i) *If $G \notin \mathfrak{H}$, then G has an $\mathfrak{H}$-critical subgroup.*

(ii) $\mathfrak{H} = E_\Phi Q R_0 \mathrm{Pr}(\mathfrak{H})$ *with* $\mathrm{Pr}(\mathfrak{H}) = \mathfrak{H} \cap \mathfrak{P}$. *Here, $\mathfrak{P}$ denotes the class of all primitive groups.*

(iii) $\mathfrak{H} = E_\Phi \mathfrak{f}$ *for some formation* $\mathfrak{f}$.

(2.4) DEFINITION [1]. Let $\mathfrak{H}$ be a Schunck class of the form $\mathfrak{H} = E_\Phi \mathfrak{f}$ for some formation $\mathfrak{f}$ and let G be a group. A subgroup D of G is an $\mathfrak{H}$-normalizer of G, if there exists a chain of subgroups:

$$D = H_n \leq H_{n-1} \leq \cdots \leq H_1 \leq H_0 = G \tag{1}$$

such that H_i is an $\mathfrak{H}$-critical subgroup of H_{i-1} ($i = 1, \ldots, n$) and such that H_n contains no $\mathfrak{H}$-critical subgroup.

If $G \in \mathfrak{H}$, we interpret the definition to mean $D = G$. The condition on H_n is equivalent to $D \in \mathfrak{H}$.

We say that a subgroup D of G is an $\mathfrak{H}$-normalizer of G *of type* 1 if there exists a chain (1) such that H_i is a maximal subgroup of H_{i-1} of type 1 for every i.

Denote by $\mathrm{Nor}_{\mathfrak{H}}(G)$ the set of all $\mathfrak{H}$-normalizers of G and by $\mathrm{Nor}_{\mathfrak{H}}(G)_1$ the set (possibly empty) of all $\mathfrak{H}$-normalizers of G of type 1.

The reader is referred to [4], [5], [9], [10] for definitions and basic results in the theory of formations and Schunck classes and to [9] for the basic properties of the primitive groups. The notation is standard and can be found mainly in [13].

3. $\mathfrak{X}$-Local formations

In this section, $\mathfrak{X}$ denotes a fixed subclass of $\mathfrak{X}_0$ subject to the following:

HYPOTHESIS. $X(\mathfrak{X}) \subseteq \mathfrak{X}$.

Moreover, $\mathfrak{f}$ will be an $\mathfrak{X}$-local formation with full and integrated local definition F.

Consider the Schunck class $\mathfrak{H} = E_\Phi \mathfrak{f}$ and let p be a prime. Define the following two classes:

$$a_1(p) = \big(G \mid \varnothing \neq \mathrm{Nor}_{\mathfrak{H}}(G)_1 \cap \mathfrak{f} \text{ and } \mathrm{Nor}_{\mathfrak{H}}(G)_1 \cap \mathfrak{f} \text{ is contained in } F(p)\big),$$

$$a_2(p) = \begin{cases} \big(G \mid \varnothing = \mathrm{Nor}_{\mathfrak{H}}(G)_1 \cap \mathfrak{f} \text{ and } G \in h(b(F(p)) \cap \mathfrak{f})\big), & \text{if } p \in X(\mathfrak{X}), \\ \big(G \mid \varnothing = \mathrm{Nor}_{\mathfrak{H}}(G)_1 \cap \mathfrak{f} \text{ and } G \in h(b_p(\mathfrak{f}))\big), & \text{if } p \notin X(\mathfrak{X}), \end{cases}$$

where $\mathrm{Nor}_{\mathfrak{H}}(G)_1 \cap \mathfrak{f}$ denotes the set of all $\mathfrak{H}$-normalizers of G of type 1 which are $\mathfrak{f}$-groups.

Denote by $a(p) = a_1(p) \cup a_2(p)$ and define $f_0(p) = Qa(p)$.

(3.1) LEMMA. *For each prime p, we have $f_0(p) \cap \mathfrak{f} = F(p)$.*

PROOF. It is clear $F(p) \subseteq f_0(p) \cap \mathfrak{f}$. Suppose that the equality does not hold and take $G \in (f_0(p) \cap \mathfrak{f})\backslash F(p)$ of minimal order. Since $G \in f_0(p) \cap \mathfrak{f} \cap b(F(p))$, there exists a group $X \in a_1(p)$ and a normal subgroup N of X such that $G \cong X/N$. Now, if $D \in \mathrm{Nor}_{\mathfrak{H}}(X)_1 \cap \mathfrak{f}$ we have that $X = DN$. Consequently, $G \in F(p)$, a contradiction.

(3.2) LEMMA. *Let p be a prime. If $\mathfrak{L}$ is a formation contained in $f_0(p)$, then $QR_0(F(p) \cup \mathfrak{L})$ is contained in $f_0(p)$.*

PROOF. It is enough to prove $R_0(F(p) \cup \mathfrak{L}) \subseteq f_0(p)$ since $f_0(p)$ is a homomorph. Suppose $R_0(F(p) \cup \mathfrak{L})$ is not contained in $f_0(p)$ and take $G \in R_0(F(p) \cup \mathfrak{L})\backslash f_0(p)$ of minimal order. Then, $G^{F(p)} \neq 1 \neq G^{\mathfrak{L}}$ and $G \notin a(p)$. Assume that $\mathrm{Nor}_{\mathfrak{H}}(G)_1 \cap \mathfrak{f}$ is non-empty and take $D \in \mathrm{Nor}_{\mathfrak{H}}(G)_1 \cap \mathfrak{f}$. By [4. Lemma 1.5], $DG^{\mathfrak{L}}/G^{\mathfrak{L}} \in \mathfrak{L}$ and $DG^{F(p)}/G^{F(p)} \in F(p)$. Consequently, $D \in R_0F(p) = F(p)$ and $G \in a(p)$, a contradiction. Therefore, we can assume that $\mathrm{Nor}_{\mathfrak{H}}(G)_1 \cap \mathfrak{f}$ is an empty set. Since $G \notin a(p)$, we have that either $G \notin h(b(F(p)) \cap \mathfrak{f})$, if $p \in X(\mathfrak{X})$ or $G \notin h(b_p(\mathfrak{f}))$, if $p \notin X(\mathfrak{X})$. Suppose $p \in X(\mathfrak{X})$ and $G \notin h(b(F(p)) \cap \mathfrak{f})$. Then, there exists a normal subgroup L of G such that $G/L \in b(F(p)) \cap \mathfrak{f}$. Now, if $L \cap G^{F(p)} = 1$, then $G \in \mathfrak{f}$, a contradiction. Consequently, we can assume that $L \cap G^{F(p)} \neq 1$ and then $G/L \cap G^{F(p)}$ lies in $f_0(p)$ by minimality of G. Therefore, $G/L \in f_0(p) \cap \mathfrak{f} = F(p)$, a contradiction. Now, if $p \notin X(\mathfrak{X})$ we argue as in [11, Lemma 3.2] to obtain the final contradiction.

(3.3) THEOREM. *Let $\mathfrak{f}$ be an $\mathfrak{X}$-local formation. Then: $\mathfrak{f}$ possesses a unique maximal $\mathfrak{X}$-local definition (as a formation) if and only if $b(\mathfrak{f})$ is $\mathfrak{X}$-wide and for each prime p, there exists a unique maximal formation, $g(p)$, contained in $a(p)$.*

In this case, the $\mathfrak{X}$-formation function g_1 defined by $g_1(p) = g(p)$ for every prime p and $g_1(E) = h(b_E(\mathfrak{f}))$ for every $E \in \mathfrak{X}' - \mathbb{P}$ is the maximal $\mathfrak{X}$-local definition.

PROOF. First, suppose that $\mathfrak{f}$ possesses a unique maximal $\mathfrak{X}$-local definition, g say. Then, $b(\mathfrak{f})$ is $\mathfrak{X}$-wide (cf. [11]). On the other hand, $g(p) \cap \mathfrak{f}$ is contained in $F(p)$. So, by [4, Lemma 1.5], $g(p)$ is contained in $a(p)$ for each prime p. Now, let $\mathfrak{L}$ be a formation contained in $a(p)$. Applying Lemma (3.2), we see that $QR_0(F(p) \cup \mathfrak{L})$ is contained in $f_0(p)$. Consider the following $\mathfrak{X}$-formation function defined by setting

$$g_2(q) = \begin{cases} QR_0(F(p) \cup \mathfrak{R}), & \text{if } p = q, \\ F(q), & \text{If } p \neq q, \end{cases}$$

and $g_2(E) = g(E)$ for every $E \in \mathfrak{X}' - \mathbb{P}$. Applying Lemma (3.1), it is not difficult to prove that $\mathfrak{f} = \mathrm{LF}_{\mathfrak{X}}(g_2)$. Since g is the unique maximal $\mathfrak{X}$-local definition of $\mathfrak{f}$, we have that $g_2(p) \subseteq g(p)$. Thus, $\mathfrak{R} \subseteq g(p)$. Consequently, $g(p)$ is the unique maximal formation contained in $a(p)$.

Conversely, suppose that $b(\mathfrak{f})$ is $\mathfrak{X}$-wide and for each prime p, there exists a unique maximal formation, $g(p)$, contained in $a(p)$. Consider the $\mathfrak{X}$-formation function g_1 defined by $g_1(p) = g(p)$ for every prime p and $g_1(E) = h(b_E(\mathfrak{f}))$ for every $E \in \mathfrak{X}' - \mathbb{P}$. We shall prove that g_1 is the maximal $\mathfrak{X}$-local definition of $\mathfrak{f}$. First of all, we need to prove that $\mathfrak{f} = \mathrm{LF}_{\mathfrak{X}}(g_1)$. Clearly, $\mathfrak{f}$ is contained in $\mathrm{LF}_{\mathfrak{X}}(g_1)$. Suppose the equality does not hold and take $G \in \mathrm{LF}_{\mathfrak{X}}(g_1) \setminus \mathfrak{f}$ of minimal order. Then, $\mathrm{Soc}(G)$ is a minimal normal subgroup of G and the composition factor of $\mathrm{Soc}(G)$ is an $\mathfrak{X}$-group. Since $G \in \mathrm{LF}_{\mathfrak{X}}(g_1)$, we have that $G/C_G(\mathrm{Soc}(G)) \in g(p) \cap \mathfrak{f} \subseteq F(p)$ or $G \in b(F(p))$ according to whether $\mathrm{Soc}(G)$ is abelian or $\mathrm{Soc}(G)$ is non-abelian ($p \in \pi(\mathrm{Soc(G)})$). In the first case, $G \in \mathfrak{f}$ and in the second one, G is an $\mathfrak{X}$-dense group. In both cases, we have a contradiction. Therefore, $\mathfrak{f} = \mathrm{LF}_{\mathfrak{X}}(g_1)$.

On the other hand, let j be an $\mathfrak{X}$-formation function such that $\mathfrak{f} = \mathrm{LF}_{\mathfrak{X}}(j)$. Applying again [4, Lemma 1.5], it is easy to prove that $j(p) \subseteq a(p)$, for every prime p. Therefore, $j(p) \subseteq g_1(p)$, for every prime p. Moreover, it is clear that $j(E)$ is contained in $g_1(E)$ for every $E \in \mathfrak{X}' - \mathbb{P}$. So, g_1 is the unique maximal $\mathfrak{X}$-local definition of $\mathfrak{f}$.

Given a group G, denote by $S_{\mathfrak{X}}(G)$ the set of all subgroups H of G such that all the composition factors of H are $\mathfrak{X}$-groups and, if $\mathfrak{R}$ is a class of groups, let $\mathfrak{R}_{\mathfrak{X}} = (G \mid \text{every } H \in S_{\mathfrak{X}}(G) \text{ is an } \mathfrak{R}\text{-group})$. It is clear that $\mathfrak{R}_{\mathfrak{X}}$ is the unique largest subgroup-closed class such that $\mathfrak{R}_{\mathfrak{X}} \cap \mathfrak{G}_{\mathfrak{X}} \subseteq \mathfrak{R}$, where $\mathfrak{G}_{\mathfrak{X}}$ is the class of all groups G such that the composition factors of G are $\mathfrak{X}$-groups.

It is known that if $\mathfrak{f}$ is an $\mathfrak{X}$-local formation, $\mathfrak{f}_{\mathfrak{X}}$ is not an $\mathfrak{X}$-local formation in general (cf. [14]). The next theorem provides precise conditions to ensure that $\mathfrak{f}_{\mathfrak{X}}$ is again an $\mathfrak{X}$-local formation.

(3.4) THEOREM. *Let $\mathfrak{f}$ be an $\mathfrak{X}$-local formation. The following statements are pairwise equivalent*:

(i) *For each primitive group G of type* 2 *in $\mathfrak{f}_{\mathfrak{X}}$ such that* $\mathrm{Soc}(G) \in \mathfrak{G}_{\mathfrak{X}}$ *and for every irreducible and faithful* $\mathrm{GF}(p)G$*-module V, $p \in \pi(\mathrm{Soc}(G))$, the corresponding semidirect product $[V]G$ is an $\mathfrak{f}_{\mathfrak{X}}$-group.*

(ii) *For each primitive group G of type* 2 *in $\mathfrak{f}_{\mathfrak{X}}$ such that* $\mathrm{Soc}(G) \in \mathfrak{G}_{\mathfrak{X}}$, *for every irreducible and faithful* $\mathrm{GF}(p)G$*-module V, $p \in \pi(\mathrm{Soc}(G))$, and for every*

$X \in S_{\mathfrak{X}}(G)$ such that $G = X\operatorname{Soc}(G)$, the semidirect product $[V]X$ is an $\mathfrak{f}$-group.

(iii) *$\mathfrak{f}_{\mathfrak{X}}$ is an $\mathfrak{X}$-local formation.*

Proof. (ii) implies (iii). Suppose $\mathfrak{f} = \mathrm{LF}_{\mathfrak{X}}(F)$, where F is the integrated and full $\mathfrak{X}$-local definition of $\mathfrak{f}$. Define $F^*(p) = F(p)_{\mathfrak{X}}$ for each prime $p \in X(\mathfrak{X})$ and $F^*(E) = F(E)_{\mathfrak{X}}$ for every $E \in \mathfrak{X}'$. It is clear that F^* is an $\mathfrak{X}$-formation function. Next, we see that $\mathfrak{f}_{\mathfrak{X}} = \mathrm{LF}_{\mathfrak{X}}(F^*)$. Assume that $\mathfrak{f}_{\mathfrak{X}}$ is not contained in $\mathrm{LF}_{\mathfrak{X}}(F^*)$ and take $G \in \mathfrak{f}_{\mathfrak{X}} \backslash \mathrm{LF}_{\mathfrak{X}}(F^*)$ of minimal order. With similar arguments to those used in [6, Theorem A], we can suppose that G is a monolithic group and $\operatorname{Soc}(G)$ is a non-abelian group with composition factor in $\mathfrak{X}$. Let p be a prime dividing the order of $N = \operatorname{Soc}(G)$ and let $X \in S_{\mathfrak{X}}(G)$. Assume that $T = XN$ is a proper subgroup of G. Since $\mathfrak{f}_{\mathfrak{X}}$ is subgroup-closed, $T \in \mathfrak{f}_{\mathfrak{X}}$ and then $T \in \mathrm{LF}_{\mathfrak{X}}(F^*)$ by minimality of G. Consequently, $T/C_T^h(N) \in F^*(p)$ where $C_T^h(N)$ is the intersection of the centralizers in T of all chief factors of T below N. Since $C_T^h(N)$ is a p-group (see [10]), we have that $XC_T^h(N)/C_T^h(N) \in S_{\mathfrak{X}}(XC_T^h(N)/C_T^h(N))$. Therefore, $XC_T^h(N)/C_T^h(N) \in F(p)$ and $X \in F(p)$. Now, if $X = G$ then $G \in F(p)$ because $\mathfrak{f}$ is $\mathfrak{X}$-local and G is primitive. Suppose that X is a proper subgroup of G and consider an irreducible and faithful $\mathrm{GF}(p)G$-module V. By (ii), the semidirect product $P = [V]X$ is an $\mathfrak{f}$-group and $X \in F(p)$. Therefore, $G \in F^*(p)$ and $G \in \mathrm{LF}_{\mathfrak{X}}(F^*)$, a contradiction.

On the other hand, taking into account that $\mathrm{LF}_{\mathfrak{X}}(F^*)$ is subgroup-closed, it is easy to see that $\mathrm{LF}_{\mathfrak{X}}(F^*)$ is contained in $\mathfrak{f}_{\mathfrak{X}}$. So, $\mathfrak{f}_{\mathfrak{X}}$ is an $\mathfrak{X}$-local formation.

Assume (iii) holds. Taking into account the $\mathfrak{X}$-local definition of $\mathfrak{f}_{\mathfrak{X}}$, it is clear that if G is a primitive group of type 2 in $\mathfrak{f}_{\mathfrak{X}}$ and $\operatorname{Soc}(G) \in \mathfrak{G}_{\mathfrak{X}}$, then the semidirect product $[V]G$ is an $\mathfrak{f}_{\mathfrak{X}}$-group for every irreducible and faithful $\mathrm{GF}(p)G$-module V, $p \in \pi(\operatorname{Soc}(G))$. Hence (i) holds.

Finally, it is clear that (i) implies (ii).

(3.5) Example. Assume $\mathfrak{X} = \mathfrak{X}_0$, the class of all simple groups, and consider the saturated formation $\mathfrak{f} = (G \mid A_5 \notin Q(G))$, where A_5 is the alternating group of degree 5. If G is a primitive group of type 2 in $\mathfrak{f}_{\mathfrak{X}}$, then every subgroup of $[V]X$ is an $\mathfrak{f}$-group, for every subgroup X of G such that $G = X\operatorname{Soc}(G)$ and for every irreducible and faithful $\mathrm{GF}(p)G$-module V ($p \in \pi(\operatorname{Soc}(G))$). Consequently, by (3.4), $\mathfrak{f}_{\mathfrak{X}}$ is a saturated formation.

4. Schunck classes and formations

(4.1) Definitions. (a) [9] Let H/K be a chief factor of G. Denote:

$$[H/K]^*G = \begin{cases} [H/K]\,(G/C_G(H/K)) & \text{if } H/K \text{ is abelian,} \\ G/C_G(H/K) & \text{if } H/K \text{ is non-abelian.} \end{cases}$$

The primitive group $[H/K]^*G$ is the *monolithic primitive group associated with the chief factor H/K of G.*

Note that if H/K is a non-Frattini chief factor of G and M is a monolithic maximal subgroup of G supplementing H/K in G, then $G/M_G \cong [H/K]^*G$.

(b) Given a Schunck class $\mathfrak{H}$, a chief factor H/K of a group G is said to be *$\mathfrak{H}$-central* in G if $[H/K]^*G \in \mathfrak{H}$ and *$\mathfrak{H}$-eccentric* otherwise.

Given a group G, define $e(G) = \{ [F] (\mathrm{Aut}_G(F) \mid F \text{ is a chief factor of } G \}$.

The following theorem is an improvement of the main result in [3], which is generalized in [12].

(4.2) THEOREM. *Let G be a group and let N be a normal subgroup of G such that $N \cap \Phi(G) = 1$. Denote by $X_1, \dots, X_s$ all monolithic primitive groups associated with the chief factors of G below N. Then: $QR_0(G) = QR_0(X_1, \dots, X_s, QR_0(G/N))$. Moreover, $QR_0(X_1, \dots, X_s, QR_0(G/N)) = R_0(X_1, \dots, X_s, QR_0(G/N))$.*

PROOF. Let $\mathfrak{K}$ denote the class $QR_0(X_1, \dots, X_s, QR_0(G/N))$. It is clear that $\mathfrak{K} \subseteq QR_0(G)$.

Let D be an $E_\Phi\mathfrak{K}$-normalizer of G. Since $G/N \in E_\Phi\mathfrak{K}$, we have that $G = DN$. On the other hand, all chief factors of G below N are $E_\Phi\mathfrak{K}$-central. Applying [1, Theorem 3.4], we see that D covers every chief factor of G below N. Therefore, D covers N and $G = D \in E_\Phi\mathfrak{K}$. Since G/N and $G/\Phi(G)$ are $\mathfrak{K}$-groups and $\mathfrak{K}$ is a formation, we have that $G \in \mathfrak{K}$.

Finally, taking into account that the class of all groups H such that every chief factor of H below $H^{\mathfrak{K}}$ is non-Frattini is a formation, it is rather easy to see the equality $QR_0(X_1, \dots, X_s, QR_0(G/N)) = R_0(X_1, \dots, X_s, QR_0(G/N))$ holds.

Consider p is prime and let G be the cyclic group of order p^2. The Frattini subgroup of G is a cyclic group of order p and $G \notin QR_0(\Phi(G))$. Consequently, the hypothesis on N in the above theorem is essential.

(4.3) DEFINITION. Let $\mathfrak{H}$ be a Schunck class and let p be a prime. Denote $f(p) = Q(G/C_G(H/K) \mid G \in \mathfrak{H}$ and H/K is a chief factor of G whose order is divisible by $p)$.

A group $G \in b(\mathfrak{H})$ is said to be *dense* with respect to $\mathfrak{H}$ if $G \in b(f(p))$ for each prime $p \in \pi(\mathrm{Soc}(G))$. Further, $b(\mathfrak{H})$ is said to be *solubly wide* if it does not contain a dense primitive group of type 1.

(4.2) REMARK. If $\mathfrak{H}$ is a Schunck class with solubly wide boundary, then $\mathfrak{H} \subseteq g(\mathfrak{H})$ where $g(\mathfrak{H}) = (G \mid \text{every Frattini chief factor of } G \text{ is } \mathfrak{H}\text{-central in } G)$.

PROOF. Suppose that $\mathfrak{H}$ is not contained in $g(\mathfrak{H})$ and let G be a group in $\mathfrak{H}\backslash g(\mathfrak{H})$. There exists a Frattini chief factor H/K of G such that $T = [H/K]^*G \notin \mathfrak{H}$. It is clear that T is a primitive group of type 1 in the boundary of $\mathfrak{H}$ which is dense with respect to $\mathfrak{H}$, a contradiction.

(4.3) THEOREM. *Let $\mathfrak{H}$ be a Schunck class. Then: $\mathfrak{H}$ is a saturated formation if and only if $b(\mathfrak{H})$ is solubly wide and consists of monolithic groups.*

PROOF. Suppose, first, $\mathfrak{H}$ is a Schunck class such that $b(\mathfrak{H})$ is solubly wide and consists of monolithic groups. We split the proof into two steps.

Step 1. $\mathfrak{H} = E_\Phi \mathfrak{f}$ for some formation $\mathfrak{f}$.

We claim that if a group H is not in $\mathfrak{H}$, then H has an $\mathfrak{H}$-critical subgroup. Suppose not and, among the groups not satisfying the above statement, we choose a group G of minimal order. It is clear that $\Phi(G) = 1$. In the case $G \in b(\mathfrak{H})$ we should have $G = M\operatorname{Soc}(G) = MF\#(G)$ for some monolithic maximal subgroup M of G. Therefore, M is $\mathfrak{H}$-critical in G, a contradiction. Hence, there exists a minimal normal subgroup N of G such that $G/N \notin \mathfrak{H}$. Let M/N be an $\mathfrak{H}$-critical subgroup of G/N. If $F^* = F\#(G \operatorname{mod} N)$, we have that $F\#(G) = F^* \cap C_G^*(N)$, where $C_G^*(N) = NC_G(N)$. Now, N is an $\mathfrak{H}$-central chief factor of G. Thus, $G/C_G^*(N)$ is an $\mathfrak{H}$-group. Since G does not contain $\mathfrak{H}$-critical subgroups, we have that $F\#(G) \leq M$. Let H/K be a chief factor of G such that $F\#(G) \leq K < H \leq F^*$, $K \leq M$ and $G = MH$. Then, the primitive group G/M_G is isomorphic to $[H/K]^*G$. On the other hand, $HC_G^*(N)/KC_G^*(N)$ is a chief factor of G which is G-isomorphic to H/K. Since $G/C_G^*(N)$ is an $\mathfrak{H}$-group, every chief factor of $G/C_G^*(N)$ is $\mathfrak{H}$-central in $G/C_G^*(N)$. This means that $HC_G^*(N)/KC_G^*(N)$ is $\mathfrak{H}$-central in G. Therefore, G/M_G is an $\mathfrak{H}$-group, a contradiction. Thus, if a group H is not in $\mathfrak{H}$, then H has a $\mathfrak{H}$-critical subgroup. Applying (2.3), we have $\mathfrak{H} = E_\Phi \mathfrak{f}$ for some formation $\mathfrak{f}$.

Step 2. $\mathfrak{H} = f(\mathfrak{H})$, where $f(\mathfrak{H}) = (G \mid e(G) \subseteq \mathfrak{H})$ is a formation.

Combination of a result of Barnes and Kegel [2] with step 1 shows that $\mathfrak{H} = E_\Phi(\mathfrak{H})$. next, we prove that $f(\mathfrak{H})$ is saturated. Let G be a group such that $G/\Phi(G) \in f(\mathfrak{H})$. Since G is an $\mathfrak{H}$-group and $\mathfrak{H} \subseteq g(\mathfrak{H})$ by (4.2), every chief factor of G below $\Phi(G)$ is $\mathfrak{H}$-central in G. Therefore, $e(G) \subseteq \mathfrak{H}$ and $G \in f(\mathfrak{H})$.

Conversely, if $\mathfrak{H}$ is a saturated formation, it is clear that $b(\mathfrak{H})$ consists of monolithic groups and, because of the celebrated Gaschütz–Lubeseder–Schmid theorem, is solubly wide.

(4.4) Example. Every Schunck class whose boundary consists of primitive groups of type 2 is a saturated formation.

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